On certain permutation representations of the braid group. Part II

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Abstract

In [1] we find certain finite homomorphic images of Artin braid group into appropriate symmetric groups, which *a posteriori* are extensions of the symmetric group on n letters by an abelian group. The main theorem of this paper characterizes completely the extensions of this type that are split.

Key words: Artin braid group, permutation representation, split extension

1 Introducton

This paper is a natural continuation of [1] and we use freely the terminology and notation used there.

In [1, Theorem 9,(iii)] we prove that the braid-like permutation group $B_n(\sigma)$ is an extension of the symmetric group Σ_n by the abelian group $A_n(q)$ from [1, (3)], and, moreover, give a sufficient condition for this extension to split. Theorem 2 is the final version of [1, Theorem 9, (iii)], and proves a necessary and sufficient condition for $B_n(\sigma)$ to be a semi-direct product of the symmetric group Σ_n and the abelian group $A_n(q)$.

2 The main theorem (continued)

Making use of [1, Theorem 9,(iii)], we have that the group $B_n(\sigma)$ is an extension of the symmetric group Σ_n by the abelian group $A_n(q)$ from [1, (3)], where $q_2 = \frac{q}{2}$, and the corresponding monodromy homomorphism is defined in [1, Proposition 11].

Any (set-theoretic) section $\rho\colon \Sigma_n \to B_n(\sigma a)$ of the surjective homomorphism π from the short exact sequence in the proof of [1, Theorem 9,(iii)] produces an

element $(a_1, \dots, a_{n-1})$ of the direct sum $\coprod_{s=1}^{n-1} A_n(q)$ via the rule $\rho(\theta_s) = \sigma_s a_s$, $s = 1, \dots, n-1$.

The next lemma is proved in a more general situation. Let A be a (left) Σ_n-module, and let

$$m\colon \Sigma_n \to Aut(A),\ \theta_s \mapsto \iota_s,\ s = 1, \dots, n-1,$$

be the corresponding structure homomorphism. Let us set

$$J_s = \iota_s + 1,\ s = 1, \dots, n-1,$$

$$I_{r,r+1} = \iota_{r,r+1} + \iota_r + \iota_{r+1},\ r = 1, \dots, n-2,$$

where $\iota_{r,r+1} = \iota_r \iota_{r+1} \iota_r = \iota_{r+1} \iota_r \iota_{r+1}$. Note that J_s and $I_{r,r+1}$ are endomorphisms of the abelian group A.

Let G be a braid-like group, let $g_1, \dots, g_{n-1}$ be a generating set of G that satisfies the braid relations, and let $\gamma\colon B_n \to G$ be the corresponding surjective homomorphism. Let $\nu\colon B_n \to \Sigma_n$ be the analogous surjective homomorphism, corresponding to the set of generators $\theta_1 \dots, \theta_{n-1}$ of the symmetric group Σ_n. Moreover, let G be an extension of Σ_n by a Σ_n-module A,

$$0 \longrightarrow A \xrightarrow{i} G \xrightarrow{\pi} \Sigma_n \longrightarrow 1, \tag{1}$$

where $\nu = \pi \circ \gamma$. We identify A with the normal abelian subgroup $i(A) \leq G$ and for any $s = 1, \dots, n-1$ denote the elements $g_s^2 \in A$ by f_s.

Lemma 1 *The extension (1) splits if and only if the linear system*

$$\begin{cases} J_s(a_s) = -f_s \\ s = 1, \dots, n-1 \\ I_{r,r+1}(a_r - a_{r+1}) = 0 \\ r = 1, \dots, n-2, \end{cases} \tag{2}$$

has a solution $a_1, \dots, a_{n-1} \in A$.

Proof: The extension (1) splits if and only if there exists a section ρ of π, which is a homomorphism of groups. By [2, Theorem 4.1], this comes to showing that there exist $a_1, \dots, a_{n-1} \in A$, such that the elements $g_s a_s$, $s = 1, \dots, n-1$, of the group G are involutions, and satisfy the braid relations. The equalities $(g_s a_s)^2 = 1$ are equivalent to

$$g_s^2 \iota_s(a_s) a_s = 1. \tag{3}$$

The equalities $g_r a_r g_{r+1} a_{r+1} g_r a_r = g_{r+1} a_{r+1} g_r a_r g_{r+1} a_{r+1}$ are equivalent to

$$g_r g_{r+1} g_r (\iota_r \iota_{r+1})(a_r) \iota_r(a_{r+1}) a_r = g_{r+1} g_r g_{r+1} (\iota_{r+1} \iota_r)(a_{r+1}) \iota_{r+1}(a_r) a_{r+1}.$$

Since g_s, $s = 1, \dots, n-1$, satisfy the braid relations, we have

$$(\iota_r \iota_{r+1})(a_r) \iota_r(a_{r+1}) a_r = (\iota_{r+1} \iota_r)(a_{r+1}) \iota_{r+1}(a_r) a_{r+1}. \tag{4}$$

Writing (3) and (4) additively, we have that the elements $a_1, \ldots, a_{n-1} \in A$ satisfy the linear system

$$\begin{cases} J_s(a_s) = -f_s, \\ s = 1, \ldots, n-1 \\ (\iota_r \iota_{r+1})(a_r) + \iota_r(a_{r+1}) + a_r = (\iota_{r+1}\iota_r)(a_{r+1}) + \iota_{r+1}(a_r) + a_{r+1}, \\ r = 1, \ldots, n-2. \end{cases}$$

After substituting a_r and a_{r+1} from the first group equations into the second one for $r = 1, \ldots, n-2$, we obtain the linear system (2).

Now, we introduce some notation. Let

$$\mathbb{Z}^n \to (\mathbb{Z}/(q))^n, \ a \mapsto \bar{a},$$

be the canonical surjective homomorphism of abelian groups. Note that the elements $\bar{f}_1,\ldots,\bar{f}_{n-1}$, $\bar{f}_n$, form a basis for the $\mathbb{Z}/(q)$-module $(\mathbb{Z}/(q))^n$. The elements $\bar{f}_1,\ldots,\bar{f}_{n-1}$, $\bar{g}_1$, $\ldots$, $\bar{g}_{n-2}$, generate $A_n(\sigma)$ as a subgroup of $(\mathbb{Z}/(q))^n$, and the group $A_n(\sigma)$ contains the elements $\bar{h}_1,\ldots,\bar{h}_n$. Moreover, there exists an isomorphism of abelian groups

$$A_n(\sigma) \simeq \mathbb{Z}\bar{f}_1/(q)\bar{f}_1 \coprod \cdots \coprod \mathbb{Z}\bar{f}_{n-1}/(q)\bar{f}_{n-1} \coprod \mathbb{Z}\bar{h}_n/(q_2)\bar{h}_n.$$

On the other hand, the group $(\mathbb{Z}/(q))^n$ has a structure of Σ_n-module, obtained via the isomorphism

$$(\mathbb{Z}/(q))^n \to \langle \tau \rangle^{(n)},$$

and $A_n(\sigma)$ is its Σ_n-submodule. Throughout the end of the paper we consider J_s and $I_{r,r+1}$ as endomorphisms of the $\mathbb{Z}/(q)$-module $(\mathbb{Z}/(q))^n$.

Theorem 2 *The extension from [1, Theorem 9, (iii)] splits if and only if* 4 *does not divide* q.

Proof: We consider several cases.

Case 1. q is an odd number. We set $a_s = -\frac{1}{2}\bar{f}_s$, $s = 1, \ldots, n-1$, where $\frac{1}{2}$ is taken modulo q. Then the first $n-1$ equations of (2) are satisfied. Further, for any $r = 1, \ldots, n-2$, we have

$$I_{r,r+1}(a_r - a_{r+1}) = -\frac{1}{2} I_{r,r+1}(\bar{f}_r - \bar{f}_{r+1}),$$

and

$$I_{r,r+1}(\bar{f}_r) = I_{r,r+1}(\bar{f}_{r+1}) = \bar{f}_r + \bar{f}_{r+1} + \bar{g}_r.$$

Therefore, the last $n-2$ equations of (2) are satisfied, too. This solution of the linear system (2) is a particular case of the solutions, used in the proof of [1, Theorem 9, (iii)].

Case 2. $q \equiv 2 \pmod 4$.

Then $q = 2q_2$, where q_2 is an odd number. Let us consider the linear system (2) over the $\mathbb{Z}/(q)$-module $(\mathbb{Z}/(q))^n$. We are looking for solutions of (2) in the abelian group $A_n(\sigma)$ considered as a $\mathbb{Z}/(q)$-submodule of $(\mathbb{Z}/(q))^n$. The reduction of $A_n(\sigma)$ modulo q_2 is the $\mathbb{Z}/(q_2)$-module $(\mathbb{Z}/(q_2))^n$, and in accord with Case 1, the reduction of the linear system (2) modulo q_2 is consistent. Now, let us reduce the system (2) modulo 2. The reduction of $A_n(\sigma)$ modulo 2 is the $\mathbb{Z}/(2)$-linear space $M = \mathbb{Z}/(2))^{n-1}$, and

$$\begin{cases} a_r = \sum_{t=r}^{n-2} \bar{f}_t \\ r = 1, \ldots, n-3 \\ a_{n-2} = \bar{f}_{n-1} \\ a_{n-1} = \bar{f}_{n-2}. \end{cases}$$

is a solution of the linear system (2) in M. Therefore the reduction of the linear system (2) modulo 2 is consistent over $A_n(\sigma)$. Thus, the linear system (2) over the $\mathbb{Z}/(q)$-module $(\mathbb{Z}/(q))^n$ is consistent, too.

Case 3. $q \equiv 0 \pmod 4$.

Let $a = \sum_{t=1}^{n-1} x_t \bar{f}_t + y\bar{h}_n$ be a generic element of the abelian group $A_n(\sigma)$. We have

$$J_1(a) = (2x_1 + x_2)\bar{f}_1 + 2\sum_{t=3}^{n-1}[(-1)^{t-1}x_2 + 2x_t]f_t + [(-1)^{n-1}x_2 + 2y]\bar{h}_n,$$

and the equality $J_1(a) = -f_1$ yields $2x_1 + x_2 + 1 \equiv 0 \pmod q$, and $2(-1)^{n-1}x_2 + 4y \equiv 0 \pmod q$, which is a contradiction.

References

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[2] Ch. Kassel, V. Turaev, *Braid Groups*, Springer, 2008.